HOSSAM GHANEM

(13) 2.5 *Discontinuous Functions* (A)

The types of discontinuity

Removable discontinuity

$$f(a) \neq \lim_{x\to a} f(x)$$

OR

$$\lim_{x\to a} f(x) = L \ \ \& \ \ f(a) \text{ Not Definite}$$

Example

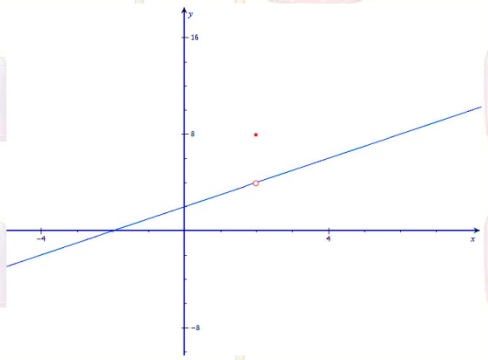
$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & , \quad x \neq 2 \\ 8 & , \quad x = 2 \end{cases}$$

$f(2) = 8$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4$$

$$f(2) \neq \lim_{x\to 2} f(x)$$

Removable discontinuity



Jumping discontinuity

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

Example

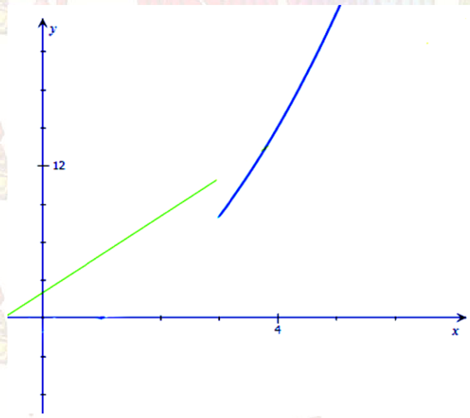
$$f(x) = \begin{cases} x^2-1 & , \quad x \leq 3 \\ 3x+2 & , \quad x > 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 2} x^2 - 1 = 9 - 1 = 8$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 2} 3x + 2 = 9 + 2 = 11$$

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

Jumping discontinuity



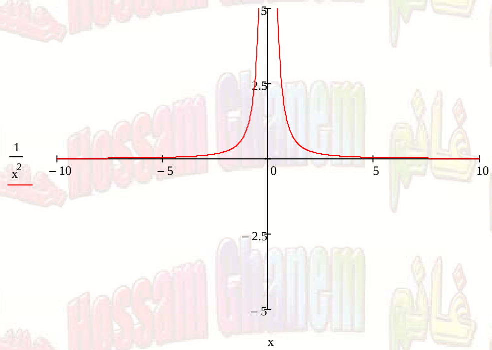
Infinite discontinuity

$$\lim_{x\to a^-} f(x) = \pm\infty \quad OR \quad \lim_{x\to a^+} f(x) = \pm\infty$$

Example $f(x) = \dfrac{1}{x^2}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty$$

Infinite discontinuity



Example 1

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[5 pts.]: The graph of a function $y = f(x)$ is given below

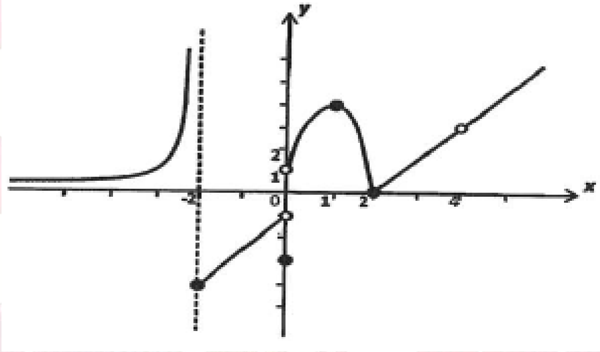

Figure 1:

Use the graph to answer the following questions. Justify your answers.

(a) Find $\lim\limits_{x \to -\infty} f(x)$

(b) What type of discontinuity does f have at $x = -2$?

(c) What type of discontinuity does f have at $x = 0$?

(d) What type of discontinuity does f have at $x = 4$?

(e) Is f differentiable at $x = 2$?

Solution

(a) $\lim\limits_{x \to -\infty} f(x) = 0$

(b) $\lim\limits_{x \to -2^-} f(x) = -\infty$ Infinite discontinuity

(c) $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$ Jumping discontinuity

(d) $f(4) \neq \lim\limits_{x \to 4} f(x)$ Removable discontinuity

(e) $f'(2^-) \neq f'(2^+)$ f not differentiable

Example 2 38 March 31, 2004	Let $f(x) = \dfrac{\lvert x+2\rvert}{\sqrt{x}\,(x^2+x-2)}$ Classify the discontinuities of f as removable , jump , or infinite.

Solution

$D_f = (0,\infty)$

$$f(x) = \frac{\lvert x+2\rvert}{\sqrt{x}\,(x^2+x-2)} = \frac{x+2}{\sqrt{x}\,(x+2)(x-1)} = \frac{1}{\sqrt{x}\,(x-1)}$$

at $x = 0$

$f(0)$ $N.D$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{\sqrt{x}\,(x-1)} \qquad D.N.E$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{\sqrt{x}\,(x-1)} = \infty$$

$\therefore$ f has infinite discont. $at\ x = 0$

at $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+2}{\sqrt{x}\,(x+2)(x-1)} = \lim_{x\to 1^-} \frac{1}{\sqrt{x}\,(x-1)} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{\sqrt{x}\,(x-1)} = \infty$$

$f(1)$ $N.D$

$\therefore$ f has infinite discont $at\ x = 1$

Example 3 48 March 25, 2008 A	Let $f(x) = \dfrac{\mid x \mid (x-3)}{x^3 - 9x}$ Classify the discontinuities of f as removable , jump , or infinite.

Solution

$x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$

$$f(x) = \frac{\mid x \mid (x-3)}{x(x-3)(x+3)}$$

at $x = -3$

$f(-3) =$ $N.D$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{-x(x-3)}{x(x-3)(x+3)} = \lim_{x\to -3^-} \frac{-1}{(x+3)} = \infty$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{-1}{(x+3)} = -\infty$$

$\therefore$ f has infinite discont at $x = -3$

at $x = 0$

$f(0)$ $N.D$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x(x-3)}{x(x-3)(x+3)} = \lim_{x\to 0^-} \frac{-1}{(x+3)} = -\frac{1}{3}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x(x-3)}{x(x-3)(x+3)} = \lim_{x\to 0^-} \frac{1}{(x+3)} = -\frac{1}{3}$$

$\therefore$ f has jump discont. at $x = 0$

at $x = 3$

$f(3)$ $N.D$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{\mid x \mid (x-3)}{x(x-3)(x+3)} = \lim_{x\to 3^-} \frac{x(x-3)}{x(x-3)(x+3)} = \lim_{x\to 3^-} \frac{1}{(x+3)} = \frac{1}{6}$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{1}{(x+3)} = \frac{1}{6}$$

$\therefore$ f has Removable discontinuity at $x = 3$

Example 4 49 July 5, 2008	Let $f(x) = \dfrac{\lvert x-1\rvert(x-2)}{(x-4)(x^2-3x+2)}$ Find all points of discontinuities of f and Classify each discontinuity as removable , jump , or infinite

Solution

$$f(x) = \frac{\lvert x-1\rvert(x-2)}{(x-4)(x^2-3x+2)} = \frac{\lvert x-1\rvert(x-2)}{(x-4)(x-1)(x-2)}$$

at $x = 1$

$f(-1)$ $N.D$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-(x-1)(x-2)}{(x-4)(x-1)(x-2)} = \lim_{x\to 1^-} \frac{-1}{(x-4)} = \frac{-1}{-3} = \frac{1}{3}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1)(x-2)}{(x-4)(x-1)(x-2)} = \lim_{x\to 1^-} \frac{1}{(x-4)} = \frac{1}{-3} = -\frac{1}{3}$$

$f(1)$ $N.D$

$\therefore$ f has jump discont. $at\ x = 1$

$at\ x = 2$

$f(2)$ $N.D$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-1)(x-2)}{(x-4)(x-1)(x-2)} = \lim_{x\to 1^-} \frac{1}{(x-4)} = \frac{1}{2}$$

$\therefore$ f has removable discont. at $x = 2$

at $x = 4$

$f(4)$ $N.D$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{1}{(x-4)} = -\infty$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{1}{(x-4)} = \infty$$

f has infifite discont. at $x = 4$

Example 5 31 October 31, 2000	Classify the discontinuities of f as removable , jump , or infinite , where $$f(x)=\frac{(x+1)\lvert x-2\rvert\sqrt{\lvert x-1\rvert}}{x^3-2x^2-x+2}$$

Solution

$x^3-2x^2-x+2=x^2(x-2)-(x-2)=(x-2)(x^2-1)=(x-2)(x-1)(x+1)$

$$f(x)=\frac{(x+1)\lvert x-2\rvert\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}$$

at $x=-1$

$f(-1)$ $N.D$

$$\lim_{x\to-1}f(x)=\lim_{x\to-1}\frac{-(x+1)(x-2)\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}=\lim_{x\to-1}\frac{-\sqrt{\lvert x-1\rvert}}{x-1}=\frac{-\sqrt{2}}{-2}=\frac{1}{\sqrt{2}}$$

$\therefore$ f has removable discont. at $x=-1$

at $x=1$

$f(1)$ $N.D$

$$\lim_{x\to1^-}f(x)=\lim_{x\to1^-}\frac{-(x+1)(x-2)\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}=\lim_{x\to1^-}\frac{-\sqrt{1-x}}{-(1-x)}=\lim_{x\to1^-}\frac{-1}{-\sqrt{1-x}}=\infty$$

$$\lim_{x\to1^+}f(x)=\lim_{x\to1^+}\frac{-(x+1)(x-2)\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}=\lim_{x\to1^+}\frac{-\sqrt{x-1}}{x-1}=\lim_{x\to1^+}\frac{-1}{\sqrt{x-1}}=-\infty$$

$\therefore$ f has infinite discont. at $x=1$

at $x=2$

$f(2)$ $N.D$

$$\lim_{x\to2^-}f(x)=\lim_{x\to2^-}\frac{-(x+1)(x-2)\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}=\lim_{x\to2^-}\frac{-\sqrt{\lvert x-1\rvert}}{(x-1)}=\frac{-1}{1}=-1$$

$$\lim_{x\to2^+}f(x)=\lim_{x\to2^+}\frac{(x+1)(x-2)\sqrt{\lvert x-1\rvert}}{(x-2)(x-1)(x+1)}=1$$

$\therefore$ f has jump discont. at $x=2$

Example 6
60 *October* 31 , 2011

. (5 points) Find the x −coordinate of the points at which the function f is discontinuous , where

$$g(x) = \begin{cases} \dfrac{2x+1}{x+3} & \text{if} \quad x < 0, \\ \dfrac{x^2-1}{x^2+x-2} & \text{if} \quad x > 0 \end{cases}$$

Classify the types of discontinuity of f as removable , jump , infinite .

Solution

$$g(x) = \begin{cases} \dfrac{2x+1}{x+3} & \text{if} \quad x < 0, \\ \dfrac{(x-1)(x+1)}{(x+2)(x-1)} & \text{if} \quad x > 0 \end{cases}$$

at $x = -3$

$f(-3)$ N.D

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{2x+1}{x+3} = \infty$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{2x+1}{x+3} = -\infty$$

$\therefore$ f has infinite discont. at $x = -3$

at $x = 0$

$f(0)$ N.D

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2x+1}{x+3} = \frac{1}{3}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2-1}{(x+2)(x-1)} = \lim_{x\to 0^-} \frac{-1}{(2)(-1)} = \frac{1}{2}$$

$\therefore$ f has jump discont. at $x = 0$

at $x = 1$

$f(1)$ *N.D*

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x+2)(x-1)} = \lim_{x\to 1} \frac{x+1}{x+2} = \frac{2}{3}$$

$\therefore$ f has removable discont . at $x = 1$

Homework

1 40 October 28, 2004 A	Find the x-coordinates of the points at which the function f is discontinuous Where $f(x)=\dfrac{x^3-8}{x^2-5x+6}$ Classify the type of discontinuity of f as removable , jump , or infinite
2 29 Feb. 24 ,2000	Find the x-coordinates of the points at which the function f is discontinuous Where $f(x)=\dfrac{x^3-1}{x^2+2x-3}$ Classify the type of discontinuity of f as removable , jump , or infinite
3 42 March 29, 2006	Find and classify the points of discontinuity, if any for the function $f(x)=\dfrac{(2x+3)(x^2-4)}{2x^2+3x-2}$
4 29 June 4. 2007	Find the $x-$coordinates of the point at which the function f is discontinuous, where $f(x)=\dfrac{x^2-x}{\lvert x\rvert(x^2-1)}$ Classify the types of discontinuity of f as removable, jump or infinite.
5 47 November 10.2007 A	Classify the discontinuities of $f(x)=\dfrac{x^2-2x}{\lvert x\rvert(x^2-x-2)}$ as removable , jump , or infinite
6 24 August 3, 2002	Classify the point of discontinuity of $f(x)=\dfrac{x(x^2-1)}{\lvert x\rvert(x^2+x-2)}$ as removable , jump , or infinite
7 43 June 28, 2008	Let $f(x)=\dfrac{(x^2-2x+1)\lvert x\rvert}{(x^2-1)x}$ Find all points of discontinuities of f and Classify each discontinuity as removable , jump , or infinite

Homework

8 58 7April 2011	[4 pts.] Find and classify (removable , jump , or infinite) the discontinuities of the function $f(x) = \dfrac{(x-2)\lvert x-1\rvert}{(x-1)(x^2-4)}$
9 56 July 10, 2010	Let $f(x) = \dfrac{(x^2-4)\sqrt{x^2+6}}{(x^3+x^2-6x)}$ a) Find and classify the discontinuities of f. (3 points) b) Find the vertical and horizontal asymptotes for the graph of f , (if any)
10 44 November 9, 2006	Find the x-coordinates of the points at which the function f is discontinuous Where $f(x) = \dfrac{x^3+8}{x^2-x-6}$ Classify the type of discontinuity of f as removable , jump , or infinite
11 41 March 30, 2005	Classify the discontinuities of $f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2} & \text{If } x < 0 \\ \dfrac{x+1}{x^2-1} & \text{If } x \geq 0 \end{cases}$ as removable , jump , or infinite

<u>10</u> 44 November 9, 2006	Find the x-coordinates of the points at which the function f is discontinuous Where $f(x) = \dfrac{x^3+8}{x^2-x-6}$ Classify the type of discontinuity of f as removable , jump , or infinite

Solution

$$f(x) = \frac{x^3+8}{x^2-x-6} = \frac{(x+2)(x^2-2x+4)}{(x+2)(x-3)}$$

$\therefore$ f is discont. at $x = -2$ & $x = 3$

at $x = -2$

$f(-2)$ $N.D$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-3)} = \lim_{x\to -2} \frac{x^2-2x+4}{x-3} = \frac{4+4+4}{-2-3} = -\frac{12}{5}$$

$\therefore$ f has removable discont. at $x = -2$

at $x = 3$

$f(3)$ $N.D$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-2x+4}{x-3} = -\infty$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2-2x+4}{x-3} = \infty$$

$\therefore$ f has infinite discont. at $x = 3$

<u>11</u> 41 March 30, 2005	Classify the discontinuities of $f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2} & \text{If } x < 0 \\ \dfrac{x+1}{x^2-1} & \text{If } x \geq 0 \end{cases}$ as removable , jump , or infinite

Solution

at $x = -2$

$f(-2)$ $N.D$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+2)(x+1)}{x+2} = \lim_{x\to -2} (x+1) = -1$$

$\therefore$ f has removable discont. at $x = -1$

at $x = 0$

$$f(0) = \frac{0+1}{0-1} = -1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+3x+2}{x+2} = \frac{2}{2} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+1}{x^2-1} = -1$$

$\therefore$ f has jump discont. at $x = 0$

at $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{(x+1)}{(x-1)(x+1)} = \lim_{x\to 1^-} \frac{1}{(x-1)} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{(x-1)} = \infty$$

$\therefore f$ has infinite discont. at $x = 1$